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Large Deflection of an Elliptic Plate under a Concentrated Load

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Introduction

THIN plates of different shapes frequently occur in many structures. Thus the study of the bending properties of a plate is imperative to a design engineer. With the increased use of strong and lightweight structures, especially in aerospace engineering, many problems of nonlinear deformation naturally arise where the supplementary stresses in the middle plane of the plate must be taken into account in deriving the differential equations of plates.

Because plates in the shape of an ellipse sometimes occur in design, large-deflection analysis of such plates has attracted many eminent research workers. Weil and Newmark[1] and Nash and Cooley[2] have investigated the large deflections of elliptic plates using von Kármán's coupled equations. Both groups have used numerical methods to obtain their solutions. Mazumdar and Jones[3] have analyzed small deformations of elliptic plates by the method of constant-deflection contour lines. The authors have extended their method to the analysis of large deflections of elliptic plates[4] by applying the well-known Berger equations.[5] Another interesting paper by Dutta[6] needs special mention because he investigated the large deflection of a clamped orthotropic elliptic plate by completely solving the differential equation for the stress function. All of these investigations are confined to uniform loading only.

A survey of the literature on nonlinear deformation of elastic plates shows that apparently no paper has been devoted to an investigation of the large deflection of an elliptic plate under a concentrated load at the center. In this Note an attempt has been made to investigate this problem by the method of constant-deflection contour lines. The numerical results obtained are shown graphically and compared.

Governing Equations and the Method of Solution

Consider a thin elastic plate of thickness h subject to a continuously distributed lateral load $q(x,y)$. Take the x,y plane to be the middle plane of the plate and direct the z axis perpendicular to that plane. The intersections between the deflected surface $z=w(x,y)$ and the plane $z=\text{const}$ yield contours which, after projection onto the $z=0$ surface, are the level curves called "lines of equal deflection." Denote the family of such curves by $u(x,y)=\text{const}$. If the boundary c of the plate is subjected to any combination of clamping and simple support, then clearly it will belong to the family of lines of equal deflection and, without loss of generality, one may consider that $u=0$ on the boundary.

Consider the equilibrium on an element Ω_u of the plate bounded by any closed contour C_u. Equating the total downward load acting on the element to the resultant upward contribution of the tractions exerted upon this portion by the remainder, one obtains

$$\oint_{c_u} V_n \mathrm{d}s - \iint_{\Omega_u} \left(q + N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2}\right) \mathrm{d}x\mathrm{d}y = 0 \tag{1a}$$

Here V_n represents the transverse reactive force which contains the shearing force Q_n and the edge rate of change of the twisting moment M_{nt} along the contour; N_x, N_y, and N_{xy} represent the membrane forces acting on a small element dxdy lying entirely within the contour C_u. Adopting Berger's approximation and substituting the well-known expressions for V_n, Q_n, and M_{nt} into Eq. (1a), as carried out in Ref. 3, one obtains

$$\frac{\mathrm{d}^3 w}{\mathrm{d}u^3} \oint_{c_u} R\mathrm{d}s + \frac{\mathrm{d}^2 w}{\mathrm{d}u^2} \oint_{c_u} F\mathrm{d}s + \frac{\mathrm{d}w}{\mathrm{d}u} \oint_{c_u} G\mathrm{d}s - \iint_{\Omega_u} (q + \alpha^2 D \nabla^2 w)\mathrm{d}x\mathrm{d}y = 0 \tag{1b}$$

where use has been made of the fact that W and its derivatives with respect to u are constant on the contour $u=\text{const}$. Here R, F, G, and t are the following expressions[3]

$$R = -Dt^{3/2}$$

$$F = -Dt^{1/2}[3u_{xx}u_x^2 + 3u_{yy}u_y^2 + u_{yy}u_x^2 + 4u_{xy}u_xu_y]$$

$$\begin{aligned} G = -Dt^{-3/2}[&u_{xxx}u_x^3u_{yyy}u_y^3 + (2-\mu)(u_{xxx}u_xu_y^2 + 4u_{xy}u_xu_y] \\ &+ u_{xyy}u_x^3 + u_{xyy}^3u_y^3 + (2\mu - 1)(u_{xyy}u_xu_y^2 + u_{xxy}u_x^2u_y) \\ &- 2(1-\mu)u_{xy}(u_xu_yu_{xx} - u_y^2u_{xy} - u_x^2u_{xy} + u_xu_yu_{yy}) \\ &+ (1-\mu)(u_{xx} - u_{yy})(u_{xx}u_y^2 - u_{yy}u_x^2)] \\ &+ 2D(1-\mu)t^{-5/2}[u_{xy}(u_x^2 - u_y^2) - u_xu_y(u_{xx} - u_{yy})]^2 \end{aligned}$$

$$t = u_x^2 + u_y^2$$

where $D = Eh^3/12(1-\mu^2)$ is the flexural rigidity, E Young's modulus, and μ Poisson's ratio. The contribution of membrane forces has been replaced by $\alpha^2 D \nabla^2 W$.[5]

As a first approximation, assume the lines of equal deflection to be as for the corresponding small-amplitude deflection problem, which are a family of similar and similarly situated ellipses. Therefore we may take[3]

$$u(x,y) = 1 - (x^2/a^2) - (y^2/b^2)$$

Calculation of the values of R, F, G, and t now gives[4]

$$R = -8D/p^3$$

$$F = -4Dp[3(x^2/a^6 + y^2/b^6) + (1-u)/a^2b^2]$$

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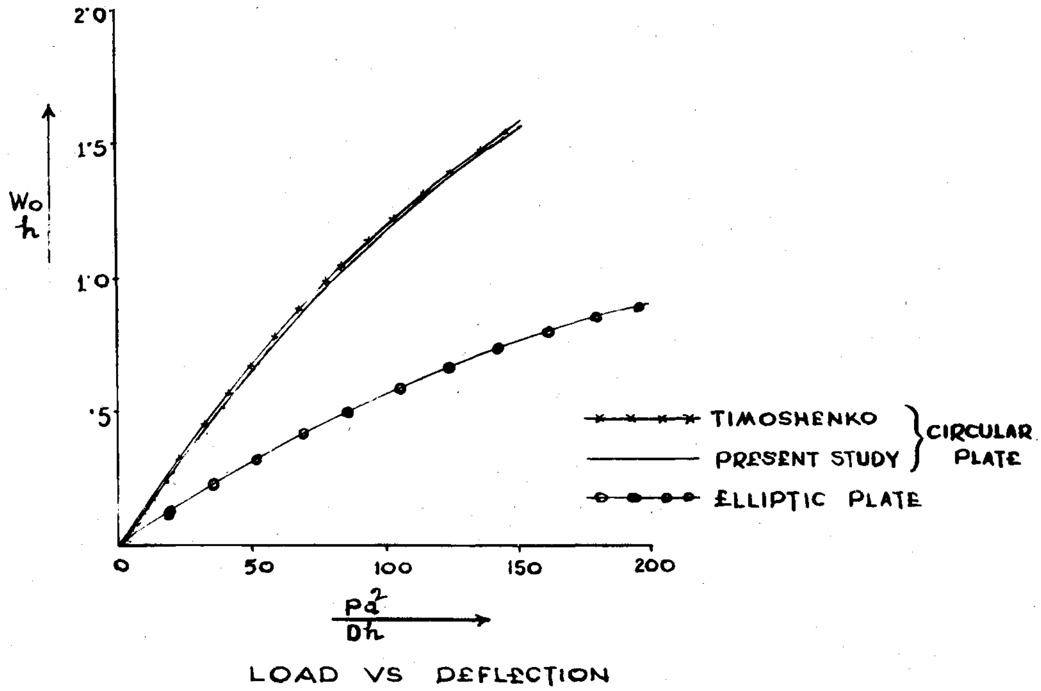

Fig. 1 Load vs deflection.

$$G=2D(1-\mu)p^5(x^2/a^4-y^2/b^4)(1/a^2-1/b^2)/a^2b^2$$

$$t=4/p^2, \text{ where } p^2=1/(x^2/a^4-y^2/b^4)$$

If the foregoing expressions are substituted into Eq. (1b) and the necessary integrations are carried out, an ordinary differential equation in terms of a new variable f, defined by

$$f^2=1-u$$

is obtained.[4]

For concentrated loading the differential equation for w is

$$\left[\frac{d^2}{df^2}+\frac{1}{f}\frac{d}{df}\right]\left[\frac{d^2}{df^2}+\frac{1}{f}\frac{d}{df}-4\gamma^2\right]w=0 \qquad (1c)$$

(except at the load point)

where

$$\gamma^2=\frac{\alpha^2a^4b^4[(1/a^2)+(1/b^2)]}{3a^4+2a^2b^2+3b^4}$$

and α^2 is determined by[4]

$$\frac{\pi\alpha^2h^2ab}{6}=\iint_S\left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right]ds \qquad (2)$$

The solution of Eq. (1) can be put in the following convenient form

$$W=c_1I_0(\lambda f)+c_2[K_0(\lambda f)+\log f]+c_3 \qquad (3)$$

where $\lambda=2\gamma$, I_0 and K_0 are the modified Bessel functions of first and second kind, c_1, c_2, c_3 constants, and $\log f$ the natural logarithm of f. For clamped edge condition

$$W=0,\quad dw/df=0 \quad \text{at} \quad f=1 \qquad (4)$$

Also discontinuity in the shearing force is equal to the load P at the center. Therefore, we have

$$\lim_{f\to 0} f\frac{d}{df}\left[\frac{1}{f}\frac{d}{df}\left(f\frac{d}{df}\right)-4\gamma^2\right]W=\frac{8Pa^3b^3}{2\pi D(3a^4+2a^2b^2+3b^4)} \qquad (5)$$

Inserting Eq. (3) into Eq. (5) one gets

$$c_2=\frac{Pab}{\pi D\alpha^2(a^2+b^2)}$$

From Eq. (4)

$$c_1=-c_2\left[\frac{1-\lambda K_1(\lambda)}{\lambda I_1(\lambda)}\right],\quad c_3=-c_1I_0(\lambda)-c_2K_0(\lambda)$$

This determines W completely.

From Eq. (2) α^2 is determined as

$$\frac{\alpha^2h^2}{6}=\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\int_0^1 f\left(\frac{dw}{df}\right)^2 df \qquad (6)$$

After evaluating the integral in Eq. (6) one obtains the following equation determining α^2.

$$\begin{aligned}\frac{\alpha^2h^2}{6}=\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\Big\{&\frac{c_1^2\lambda^2}{2}\left[I_1^2(\lambda)-I_0^2(\lambda)+\frac{2}{\lambda}I_1(\lambda)I_0(\lambda)\right]\\&+\frac{c_2^2\lambda^2}{2}\left[K_1^2(\lambda)-K_0^2(\lambda)-\frac{2}{\lambda}K_1(\lambda)K_0(\lambda)\right]\\&-c_1c_2\lambda^2[I_1(\lambda)K_1(\lambda)+I_0(\lambda)K_0(\lambda)]\\&+c_1c_2\lambda[I_1(\lambda)K_0(\lambda)-I_0(\lambda)K_1(\lambda)]+2c_1c_2I_0(\lambda)\\&+2c_2^2K_0(\lambda)-c_1c_2-\frac{c_2^2}{2}+c_2^2\left[\log\frac{\lambda}{2}+\gamma_1\right]\Big\}\end{aligned} \qquad (7)$$

where γ_1 = Euler's const.

Numerical Results

To obtain the desired deflection one has to start from Eq. (7) with assumed values of αa leading to the particular value of the load function. This value of load function finally determines W from Eq. (3). For the ellipse $a = 2b$ deflections have been calculated with $\alpha a = 1,2,3,....$ For $a = b$, the ellipse reduces to a circle. Numerical results for both circular and elliptic plates have been plotted graphically. It is clear from the graph that the results for the circular plate are in very good agreement with those in Ref. 7. The results for the elliptic plate are new.

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